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A Note on the Characteristic Cycle of the Image of the Constant Sheaf

by

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Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$ module on a complex manifold X. Then the characteristic variety of $\mathscr{M}$ is contained in $\bigcup_\alpha T^*_{X_\alpha}X$ for a certain stratification of X. Here $T^*_{X_\alpha}X$ is the cotangent bundle of X_α in X. The characteristic cycle of $\mathscr{M}$ is defined to be

$$Ch\mathscr{M}=\sum m_\alpha(\mathscr{M})\overline{T^*_{X_\alpha}X}$$

where $m_\alpha(\mathscr{M})$ is the multiplicity of $\mathscr{M}$ along $T^*_{X_\alpha}X$. In the sequel we write simply $T^*_{X_\alpha}X$ for $\overline{T^*_{X_\alpha}X}$. Let $\varphi: \tilde{X}\to Y$ be a projective map of complex manifolds and Γ_φ a submanifold of $\tilde{X}\times Y$ which is the graph of φ. We denote by $v_{\tilde{X}}$, v_Y the projections of the cotangent bundle $T^*(\tilde{X}\times Y)=T^*\tilde{X}\times T^*Y$ to its factors. Let $\mathscr{M}$ be a holonomic $\mathscr{D}_{\tilde{X}}$ module with a global good filtration. Then in Ginsburg [5] the following formula is announced:

$$\sum(-1)^i Ch\left(\int_\varphi^i \mathscr{M}\right)=(v_Y)_*(T^*_{\Gamma_\varphi}(\tilde{X}\times Y)\cdot v^*_{\tilde{X}}Ch\mathscr{M})\,.$$

In this short note we calculate $T^*_{\Gamma_\varphi}(\tilde{X}\times Y)\cdot v^*_{\tilde{X}}Ch\mathscr{M}$ for $\mathscr{M}=\mathscr{O}_{\tilde{X}}$. Let $\varphi^*T^*Y=\tilde{X}\times_X T^*Y$ be the induced bundle of T^*Y on $\tilde{X}$ and $\gamma: \varphi^*T^*Y\to T^*Y$ and $\varpi: \varphi^*T^*Y\to\tilde{X}$ the canonical projections. Denote by α_Y the canonical 1-form on T^*Y. Then $\gamma^*\alpha_Y$ may be considered to be a section of the induced bundle $\varpi^*T^*\tilde{X}$. The section $\gamma^*\alpha_Y$ determines the localized top Chern class $\mathbb{Z}[\gamma^*\alpha_Y]$ on T^*Y by Fulton [4].

THEOREM. *The characteristic cycle of the constructible sheaf* $\mathbb{R}\varphi_*\mathbb{C}_{\tilde{X}}=\mathbb{R}\varphi_*DR\mathscr{O}_{\tilde{X}}$ *is equivalent to* $\gamma_*\mathbb{Z}[\gamma^*\alpha_Y]$.

According to Sabbah [6] the assumption that the map φ is projective is superfluous. In this note we follow the intersection theory of Fulton [4]. For example, the intersection of a cycle x_i of dimension i and a cycle y_j of dimension j in a manifold X of dimension n is a homology class of Borel-Moor homology group $H_{2(n-i-j)}(|x_i|\cap|y_j|)$, where $|x_i|$, $|y_j|$ are the supports of the cycles. An analytic space is one possively with nilpotent elements in its structure sheaf and a reduced analytic space is called variety. As an application we calculate the local Euler number of a subvariety X of a manifold Y applying our theorem to a resolution of singularity $\varphi: \tilde{X}\to X\subset Y$ and the local index theorem of Brylinski, Dubson and Kashiwara [3].

1. Proof of theorem

At first we prove the theorem of Ginsburg for completeness' sake. Let $\varphi\colon \tilde{X}\to Y$ be a projective map of complex manifolds and $\mathscr{M}$ be a holonomic $\mathscr{D}_{\tilde{X}}$ module with good filtration. Denote by $\Omega_{\tilde{X}}$ the sheaf of holomorphic n forms on $\tilde{X}$, n being the dimension of $\tilde{X}$. Identifying $\tilde{X}\times_Y T^*Y$ with the conormal bundle $T^*_{\Gamma_\varphi}(\tilde{X}\times Y)$ to the graph Γ_φ of φ, we denote by ρ the restriction of the projection $\nu^*_{\tilde{X}}$ to $\tilde{X}\times_Y T^*Y$.

$$\begin{tikzcd}
 & & T^*Y \\
\tilde{X}\times_Y T^*Y\simeq T^*_{\Gamma_\varphi}(\tilde{X}\times Y) \arrow[urr, "\gamma"] \arrow[drr, "\rho"'] \arrow[rr, hook, "i"'] & & T^*(\tilde{X}\times Y) \arrow[u, "\nu^*_Y"'] \arrow[d, "\nu^*_{\tilde{X}}"'] \\
 & & T^*\tilde{X}
\end{tikzcd}$$

The support $\mathscr{H}^k$ of the cohomology sheaf $\mathscr{H}^k(R\gamma_* L\rho^*(\pi^*\Omega_{\tilde{X}}\otimes_{\mathscr{O}_{T^*\tilde{X}}} gr\mathscr{M}))$ is a Lagrangean subvariety of T^*Y, where $\pi\colon T^*\tilde{X}\to\tilde{X}$ is the projection. We denote by $[\mathscr{F}]$ the i cycle on X defined by $\mathscr{F}$ for a coherent sheaf $\mathscr{F}$ on a complex manifold X with $\dim\sup\mathscr{F}=i$ as usual. Put

$$[R\gamma_* L\rho^*(\pi^*\Omega_{\tilde{X}}\otimes_{\mathscr{O}_{T^*\tilde{X}}} gr\mathscr{M})]=\sum(-1)^k[\mathscr{H}^k(R\gamma_* L\rho^*(\pi^*\Omega_{\tilde{X}}\otimes_{\mathscr{O}_{T^*\tilde{X}}} gr\mathscr{M}))]\,.$$

Put $Z=\bigcup_k \mathscr{H}^k$. Then Z is purely of dimension $m=\dim Y$, and the cycle $[R\gamma_* L\rho^*(\pi^*\Omega_{\tilde{X}}\otimes_{\mathscr{O}_{T^*\tilde{X}}} gr\mathscr{M})]$ may be considered to be a class of the Borel-Moore homology group $H_{2m}(Z)$. For the proof of the theorem of Ginsburg the following theorem is sufficient.

THEOREM. *We have*

$$[R\gamma_* L\rho^*(\pi^*\Omega_{\tilde{X}}\otimes_{\mathscr{O}_{T^*\tilde{X}}} gr\mathscr{M})]=(\nu_Y)_*(T^*_{\Gamma_\varphi}(\tilde{X}\times Y)\cdot \nu^*_{\tilde{X}} Ch\mathscr{M})$$

where '$\cdot$' on the right hand means the refined intersection of Fulton [4] *and the equality is the one in the Borel-Moore homology group $H_{2m}(Z)$.*

Proof. It is easy to see that it suffices to prove the theorem locally on Y. Since the morphism $\nu^*_{\tilde{X}}$ is flat, we have $L\nu^*_{\tilde{X}}=\nu^*_{\tilde{X}}$. Put

$$\mathscr{F}=\nu^*_{\tilde{X}}(\pi^*\Omega_{\tilde{X}}\otimes_{\mathscr{O}_{T^*\tilde{X}}} gr\mathscr{M})\,.$$

The support S of $\mathscr{F}$ is Lagrangean and we have $[\mathscr{F}]=\nu^*_{\tilde{X}} Ch\mathscr{M}$. Put $\Lambda=T^*_{\Gamma_\varphi}(\tilde{X}\times Y)$ and let $\mathscr{I}_\Lambda$ be the defining ideal of Λ in $\mathscr{O}_{T^*(\tilde{X}\times Y)}$. The graded module $\sum_k \mathscr{I}^k_\Lambda\mathscr{F}/\mathscr{I}^{k+1}_\Lambda\mathscr{F}$ defines the coherent module on the normal bundle $T_\Lambda(T^*(\tilde{X}\times Y))$ to Λ in $T^*(\tilde{X}\times Y)$, which we denote by $\mathrm{Sp}_\Lambda(\mathscr{F})$. If $\mathscr{F}$ is a coherent sheaf on a complex manifold X, we denote by the same letter $\mathscr{F}$ the class of $\mathscr{F}$ in the Grothendieck group $K(X)$. We define $|\mathscr{F}^\bullet|\in K(X)$ for a complex $\mathscr{F}^\bullet$ by putting $|\mathscr{F}^\bullet|=\sum(-1)^k\mathscr{H}^k(\mathscr{F}^\bullet)$. Let $q\colon T_\Lambda(T^*(\tilde{X}\times Y))\to\Lambda$ be the projection and $\iota\colon \Lambda\to T_\Lambda(T^*(\tilde{X}\times Y))$ the zero section. Then by Verdier [7] we have $\mathrm{Sp}_\Lambda(\mathscr{F})=|Lq^*Li^*\mathscr{F}|$. Hence we have

$$|L\rho^*(\pi^*\Omega_{\tilde{X}}\otimes_{\mathcal{O}_{T^*\tilde{X}}} gr\mathcal{M})|=|Li^*L\nu_{\tilde{X}}^*(\pi^*\Omega_{\tilde{X}}\otimes_{\mathcal{O}_{T^*\tilde{X}}} gr\mathcal{M})|=|Li^*\mathcal{F}|=\imath^*\mathrm{Sp}_\Lambda(\mathcal{F}).$$

Denote by C the support of $\mathrm{Sp}_\Lambda(\mathcal{F})$ and $\kappa\colon C\to T_\Lambda(T^*(\tilde{X}\to Y))$ the injection. Then C is a cone in $T_\Lambda(T^*(\tilde{X}\times Y))$ and actually contained in $T_\Lambda(T^*(\tilde{X}\times Y))|\Lambda\cap S=q^{-1}(\Lambda\cap S)$. Let $i^!\colon H.(S)\to H.(S\cap\Lambda)$ be the refined Gysin homomorphism of Borel-Moor homology groups.

$$\begin{tikzcd} S\cap\Lambda \arrow[r] \arrow[d] & S \arrow[d] \\ \Lambda \arrow[r, "i"] & T^*(\tilde{X}\times Y) \end{tikzcd}$$

Then by definition we have

$$i^!([\mathcal{F}])=\imath_S^*([\mathrm{Sp}_\Lambda(\mathcal{F})])=[\mathcal{F}]\cdot\Lambda,$$

where $\imath_S$ is the restriction of $\imath$ to $S\cap\Lambda$ and $\imath_S^*$ is the associated refined Gysin homomorphism. Let $\mathcal{I}_C$ be the defining ideal of C in $\mathcal{O}_{T^*(\tilde{X}\times Y)}$ and define an element z of the Grothendieck group $K(C)$ by putting

$$z=\sum_{k\geq 0}\mathcal{I}_C^k\mathrm{Sp}_\Lambda(\mathcal{F})/\mathcal{I}_C^{k+1}\mathrm{Sp}_\Lambda(\mathcal{F}).$$

Then we have $\kappa_*(z)=\mathrm{Sp}_\Lambda(\mathcal{F})$ in $K(T_\Lambda(T^*(\tilde{X}\times Y)))$ and

$$ch(z)=[\mathrm{Sp}_\Lambda(\mathcal{F})]+\text{terms of lower dimension}.$$

Here and in the rest the Chern character '*ch*' is the localized Chern character. This means, for example, that $ch(z)\in H.(C)$. Since we have

$$ch(\mathrm{Sp}_\Lambda(\mathcal{F}))=[\mathrm{Sp}_\Lambda(\mathcal{F})]+\text{terms of lower dimension},$$

and

$$ch(\imath^*(\mathrm{Sp}_\Lambda(\mathcal{F})))=\imath^*ch(\mathrm{Sp}_\Lambda(\mathcal{F}))=\imath^*[\mathrm{Sp}_\Lambda(\mathcal{F})]+\text{terms of lower dimension},$$

we have

$$\begin{aligned} ch(|Li^*\mathcal{F}|)&=ch(\imath^*\mathrm{Sp}_\Lambda(\mathcal{F}))\\ &\equiv[\mathcal{F}]\cdot\Lambda \quad \text{mod terms of lower dimension}. \end{aligned}$$

Let $\lambda\colon Z=\bigcup\operatorname{supp}\mathcal{H}^k(R\gamma_*L\rho^*(\pi^*\Omega_{\tilde{X}}\otimes_{\mathcal{O}_{T^*\tilde{X}}} gr\mathcal{M}))\to T^*Y$ be the injection. Then the subvariety $\gamma(\Lambda\cap S)$ is contained in Z. We denote by γ_S the map of $\Lambda\cap S$ to Z induced by γ and by λ_S the restriction of λ to $\Lambda\cap S$.

$$\begin{tikzcd} \tilde{X}\times_Y T^*Y=\Lambda & \Lambda\cap S \arrow[l, "\lambda_S"'] \arrow[d, "\gamma_S"] \\ T^*Y \arrow[u, "\gamma"] & Z \arrow[l, "\lambda"'] \end{tikzcd}$$

By the Grothendieck-Riemann-Roch theorem we have

$$\begin{aligned} ch(\gamma_{s^*}\iota_S^* Z) &\equiv \gamma_{s^*} ch(\iota_S^* z) \quad \text{mod terms of lower dimension} \\ &\equiv \gamma_{s^*}(\Lambda \cdot [\mathscr{F}]) \quad \text{mod terms of lower dimension}\,. \end{aligned}$$

Since we have

$$\begin{aligned} & ch(|\, R\gamma_* L\rho^*(\pi^*\Omega_{\tilde{X}} \otimes_{\mathcal{O}_{T^*\tilde{X}}} gr\mathscr{M})\,|) \\ & \quad \equiv [R\gamma_* L\rho^*(\pi^*\Omega_{\tilde{X}} \otimes_{\mathcal{O}_{T^*\tilde{X}}} gr\mathscr{M})] \quad \text{mod terms of lower dimension}\,, \end{aligned}$$

we readily infer that

$$[R\gamma_* L\rho^*(\pi^*\Omega_{\tilde{X}} \otimes_{\mathcal{O}_{T^*\tilde{X}}} gr\mathscr{M})] = \lambda_* \gamma_{S^*}(\Lambda \cdot [\mathscr{F}])\,.$$

This means the cycle $[R\gamma_* L\rho^*(\pi^*\Omega_{\tilde{X}} \otimes_{\mathcal{O}_{T^*\tilde{X}}} gr\mathscr{M})]$ coincides with the image $\gamma_*(\Lambda \cdot [\mathscr{F}])$ of the refined intersection $\Lambda \cdot [\nu_{\tilde{X}}^* Ch\mathscr{M})]$.

Now we shall prove our theorem. Let p and q be the projections of $\tilde{X}\times Y$ to its factors. Identifying $\tilde{X}$ with the graph Γ_φ, we consider $\tilde{X}$ to be a submanifold of $\tilde{X}\times Y$ and denote by $j\colon \tilde{X}\to\tilde{X}\times Y$ the canonical injection. We may identify the restriction $q^*T^*Y|\tilde{X}$ of the induced bundle q^*T^*Y over $\tilde{X}\times Y$ to the submanifold $\tilde{X}$ with the induced bundle φ^*T^*Y. We have the commutative diagram.

$$\begin{tikzcd} q^*T^*Y \arrow[r, "\pi"] & \tilde{X}\times Y \arrow[r, "q"] & Y \\ \varphi^*T^*Y \arrow[u] \arrow[r, "\varpi"] & \tilde{X} \arrow[u, "j"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

where π is the projection. The variety φ^*T^*Y is a subvariety of q^*T^*Y. The variety $T^*(\tilde{X}\times Y)$ is identified with the induced bundle $\pi^*(p^*T^*\tilde{X})$ as varieties and hence $T^*_{\Gamma_\varphi}(\tilde{X}\times Y)$ is considered to be a subvariety of $\pi^*(p^*T^*\tilde{X})$. The image of $T^*_{\Gamma_\varphi}(\tilde{X}\times Y)$ by the projection map of the bundle $\pi^*(p^*T^*\tilde{X})\to\tilde{X}\times Y$ is contained in $\tilde{X}$. That is, the variety $T^*_{\Gamma_\varphi}(\tilde{X}\times Y)$ may be considered to be a subvariety of $\pi^*(p^*T^*\tilde{X})\,|\,\varphi^*T^*Y = \varpi^*j^*p^*T^*\tilde{X} = \varpi^*T^*\tilde{X}$.

PROPOSITION. *The subvariety $T^*_{\Gamma_\varphi}(\tilde{X}\times Y)$ is considered to be a cross section of the vector bundle $\varpi^*T^*\tilde{X}\to\varphi^*T^*Y$. In fact, letting α_Y be the canonical 1-form on T^*Y and defining the cross section $\sigma\colon \varphi^*T^*Y\to\varpi^*T^*\tilde{X}$ to be $\sigma(z) = -\gamma^*(\alpha_Y)_z$ for $z\in\varphi^*T^*Y$, where $\gamma\colon \varphi^*T^*Y\to T^*Y$ is the canonical projection and $\gamma^*(\alpha_Y)_z$ is the germ of 1-form at $z\in\varphi^*T^*Y$ which belongs to $(\varpi^*T^*\tilde{X})_z$, we have $T^*_{\Gamma_\varphi}(\tilde{X}\times Y) = \sigma(\varphi^*T^*Y)$.*

Proof. Take a point x of $\tilde{X}$ and put $y=\varphi(x)$. Let (x_ν) be a system of local coordinates of $\tilde{X}$ with center x and (y_i) one of Y with center y. Denote by (ξ_ν) (resp. (η_i)) the fibre coordinates of the cotangent bundle $T^*\tilde{X}$ (resp. T^*Y) with respect to (x_ν) (resp. (y_i)). Then the subvariety $T^*_{\Gamma_\varphi}(\tilde{X}\times Y)$ in φ^*T^*Y is defined by the equations:

$$\xi_\nu + \sum_i \eta_i \frac{\partial y_i(\varphi(x))}{\partial x_\nu} = 0 .$$

On the other hand we have

$$\gamma^*(\alpha_Y) = \sum_i \eta_i dy_i(\varphi(x)) = \sum_{i,\nu} \eta_i \frac{\partial y_i(\varphi(x))}{\partial x_\nu} dx_\nu .$$

That is, the section σ is defined by the equations:

$$\xi_\nu = -\sum_i \eta_i \frac{\partial y_i(\varphi(x))}{\partial x_\nu} .$$

We continue the proof of the theorem. Let s be the zero section of the vector bundle $T^*(\tilde{X}\times Y) = \pi^*(p^*T^*\tilde{X}) \to q^*T^*Y$. Since the characteristic cycle $Ch(\mathcal{O}_{\tilde{X}})$ is $T^*_{\tilde{X}}\tilde{X}$, we have $\nu^*_{\tilde{X}} Ch(\mathcal{O}_{\tilde{X}}) = s(q^*T^*Y)$, where $\nu^*_{\tilde{X}}$ is the projection of $T^*(\tilde{X}\times Y)$ to the factor $T^*\tilde{X}$. Hence the cycle $T^*_{\Gamma_\varphi}(\tilde{X}\times Y).\ \nu^*_{\tilde{X}} Ch(\mathcal{O}_{\tilde{X}})$ is equivalent to the cycle which is obtained as the intersection $\sigma(\varphi^*T^*Y)\cdot s(q^*T^*Y) = \sigma(\varphi^*T^*Y)\cdot s(\varphi^*T^*Y)$. Here $T^*_{\Gamma_\varphi}(\tilde{X}\times Y)\cdot \nu^*_{\tilde{X}} Ch(\mathcal{O}_{\tilde{X}})$ and $\sigma(\varphi^*T^*Y)\cdot s(\varphi^*T^*Y)$ are considered to be cycles in $\varphi^*T^*Y \cap \sigma(\varphi^*T^*Y)$. By definition (Fulton [4]) the intersection cycle $\sigma(\varphi^*T^*Y)\cdot s(\varphi^*T^*Y)$ is the localized top Chern class $\mathbb{Z}[\sigma]$ of the section σ and we have $\mathbb{Z}[-\sigma] = \mathbb{Z}[\sigma]$. Thus we have completed the proof of the theorem.

2. An application

Let X be an irreducible subvariety of dimension n of a manifold Y and $\varphi: \tilde{X} \to X \subset Y$ a resolution of singularity. Then the characteristic cycle $ChR\varphi_*\mathcal{O}_{\tilde{X}}$ is the form:

$$ChR\varphi_*\mathcal{O}_{\tilde{X}} = T^*_X Y + \sum m_j T^*_{X_j} Y ,$$

where X_j is a subvariety of X of lower dimension than n. Hence by the local index theorem of Brylinski, Dubson and Kashiwara [3] we have

$$\mathcal{X}_x(R\varphi_*\mathbb{C}_{\tilde{X}}) = (-1)^n Eu_X(x) + \sum (-1)^{n_j} m_j Eu_{X_j}(x) ,$$

where $n_j = \dim X_j$, and $Eu_X(x)$, $Eu_{X_j}(x)$ are local Euler numbers at x. If we can calculate $\mathcal{X}_x(R\varphi_*\mathbb{C}_{\tilde{X}})$ and m_j, we can calculate $Eu_X(x)$ by induction on dimension of varieties.

Example 1. Let $Y = \mathbb{C}^3$ and X the subvariety of Y defined by the equation $x^a + y^a + z^a = 0$, a being an integer with $a \geq 2$. We calculate the local Euler number $Eu_X(o)$ at the origin o. Let $\tilde{X}$ be the blow-up of X at o:

$$\tilde{X} \subset Y \times \mathbb{P}^2: \quad \lambda_i x_j = \lambda_j x_i , \quad \lambda_0^a + \lambda_1^a + \lambda_2^a = 0 ,$$

where (λ_i) is a homogeneous coordinate system of the projective $\mathbb{P}^2$. If $\lambda_0 \neq 0$, and $\lambda_2 \neq 0$, then $\left(x, \dfrac{\lambda_1}{\lambda_0}\right)$ is a local coordinate system of $\tilde{X}$ and the canonical map $\varphi: \tilde{X} \to Y$ is expressed as

$$\varphi:\left(x, \frac{\lambda_1}{\lambda_2}\right) \longrightarrow (x, y, z) = \left(x, \frac{\lambda_1}{\lambda_0}x, x\left(-1-\frac{\lambda_1}{\lambda_0}\right)^{1/a}\right).$$

Let (ξ, η, ζ) be the fibre coordinates of the cotangent bundle T^*Y and α_Y the canonical 1-form on T^*Y. Then we have

$$\gamma^*\alpha_Y = \left\{\xi + \eta\frac{\lambda_1}{\lambda_0} + \zeta\left(-1-\frac{\lambda_1}{\lambda_0}\right)^{1/a}\right\}dx + x\left\{\eta x + \zeta\frac{d}{d\frac{\lambda_1}{\lambda_0}}\left(-1-\frac{\lambda_1}{\lambda_0}\right)^{1/a}\right\}d\frac{\lambda_1}{\lambda_0},$$

where $\gamma: \varphi^*T^*Y \to T^*Y$ is the canonical projection. The component of the localized Chern class $\mathbb{Z}[\gamma^*\alpha_Y]$ which is over the exceptional curve Θ is defined by the equations:

$$\xi + \eta\frac{\lambda_1}{\lambda_0} + \zeta\left(-1-\frac{\lambda_1}{\lambda_0}\right)^{1/a} = 0, \qquad x = 0.$$

It is easy to see that the number of the points which satisfies the above equations for general (ξ, η, ζ) is a. Therefore we have

$$ChR\varphi_*\mathbb{C}_{\tilde{X}} = T^*_X Y + aT^*_o Y,$$

from which and the fact that

$$\mathscr{X}_0(\varphi_*\mathbb{C}_{\tilde{X}}) = \mathscr{X}(\Theta) = 2-(a-1)(a-2),$$

we have

$$Eu_X(o) = 2-(a-1)(a-2)-a = 2a-a^2.$$

Example 2. Let $X = \{x^2 = y^2 z\} \subset Y = \mathbb{C}^3$ be Whitney umbrella. Put

$$\tilde{X} \subset Y \times \mathbb{P}^1 : \lambda_1 x = \lambda_0 y, \qquad \lambda_0^2 - z\lambda_1^2 = 0,$$

and let $\varphi: \tilde{X} \to Y$ be the natural map. If $\lambda_1 \neq 0$, then $\left(y, \frac{\lambda_0}{\lambda_1}\right)$ is a local coordinate system of $\tilde{X}$ and φ is expressed as

$$\varphi\left(y, \frac{\lambda_0}{\lambda_1}\right) = (x, y, z) = \left(y\frac{\lambda_0}{\lambda_1}, y, \left(\frac{\lambda_0}{\lambda_1}\right)^2\right).$$

Hence we have

$$\gamma^*(\alpha_Y) = \left(\xi\frac{\lambda_0}{\lambda_1} + \eta\right)dy + \left(\xi y + 2\zeta\frac{\lambda_0}{\lambda_1}\right)d\frac{\lambda_0}{\lambda_1},$$

where α_Y is the canonical 1-form and (ξ, η, ζ) is the fibre coordinates of the cotangent bundle T^*Y. The localized top Chern class $\mathbb{Z}[\gamma^*(\alpha_Y)]$ is defined by the equations

$$\xi\frac{\lambda_0}{\lambda_1}+\eta=0\,,\qquad \xi y+2\zeta\frac{\lambda_0}{\lambda_1}=0\,,$$

which is easily seen to be an irreducible subvariety. Hence we have $ChR\varphi_*\mathcal{O}_{\tilde{X}}=T^*_XY$. On the other hand we have

$$\mathcal{X}_p(R\varphi_*\mathbb{C}_{\tilde{X}})=\mathcal{X}(\varphi^{-1}(p))=\begin{cases}1 & \text{if}\quad p=(0,0,0)\,,\\ 2 & \text{if}\quad p=(0,0,z)\,,\quad z\neq 0\,.\end{cases}$$

Hence we have

$$Eu_X(p)=\begin{cases}1 & \text{if}\quad p=(0,0,0)\,,\\ 2 & \text{if}\quad p=(0,0,z)\,,\quad z\neq 0\,.\end{cases}$$

Example 3. We consider an automorphism σ of $\mathbb{C}^n$ with coordinates (x_i) defined to be

$$\sigma:(x_i)\longrightarrow(e^{(2\pi/a)\sqrt{-1}}x_i)$$

where a is a positive integer, and the quotient space $X=\mathbb{C}^n/(\sigma)$. The image of the origin o of $\mathbb{C}^n$ is the unique singular point of X, which we denote by the same letter o. The complex space X may be considered to be a subvariety of $Y=\mathbb{C}^N$ where $N={}_n\mathrm{H}_a$ by the map

$$X=\mathbb{C}^n/(\sigma)\ni\text{class of }(x_i)\longrightarrow(y_{\alpha_1\cdots\alpha_n})=(x_1^{\alpha_1}\cdots x_n^{\alpha_n})\in\mathbb{C}^N\,,$$

where α_i are nonnegative integer with $\sum\alpha_i=a$, Put $\tilde{X}=\{\lambda_j^a\xi_i=\lambda_i^a\xi_j\}\subset\mathbb{C}^n\times\mathbb{P}^{n-1}$ where (ξ_i) is a coordinate system of $\mathbb{C}^n$ and (λ_i) is a homogeneous coordinate system of $\mathbb{P}^{n-1}$. Then $\tilde{X}$ is a submanifold and the map

$$\tilde{X}\ni(\xi_i,\lambda_i)\longrightarrow\text{class of}\left(\xi_1,\frac{\lambda_2}{\lambda_1}\xi^{1/a},\cdots,\frac{\lambda_n}{\lambda_1}\xi^{1/a}\right)\in X\,,$$

which is defined on the part where $\lambda_1\neq 0$, induces the holomorphic map $\varphi:\tilde{X}\to X$ and if we consider φ to be the map of $\tilde{X}$ to $Y=\mathbb{C}^N$, then we have

$$\varphi(\xi,\lambda)=(y_{\alpha_1\cdots\alpha_n})=\left(\xi_1\left(\frac{\lambda_2}{\lambda_1}\right)^{\alpha_2}\cdots\left(\frac{\lambda_n}{\lambda_1}\right)^{\alpha_n}\right).$$

Denote by $(\eta_{\alpha_1\cdots\alpha_n})$ the fibre coordinates of T^*Y. Then we have

$$\begin{aligned}\gamma^*(\alpha_Y)&=\sum\eta_{\alpha_1\cdots\alpha_n}dy_{\alpha_1\cdots\alpha_n}(\xi,\lambda)\\&=\sum\eta_{\alpha_1\cdots\alpha_n}\left(\frac{\lambda_2}{\lambda_1}\right)^{\alpha_2}\cdots\left(\frac{\lambda_n}{\lambda_1}\right)^{\alpha_n}d\xi_1\\&\quad+\xi_1\sum\nolimits_{k\geq 2,\alpha_k\geq 1}\eta_{\alpha_1\cdots\alpha_n}\alpha_k\left(\frac{\lambda_2}{\lambda_1}\right)^{\alpha_2}\cdots\left(\frac{\lambda_k}{\lambda_1}\right)^{\alpha_k-1}\cdots\left(\frac{\lambda_n}{\lambda_1}\right)^{\alpha_n}d\left(\frac{\lambda_k}{\lambda_1}\right).\end{aligned}$$

Now we denote by s the section of the bundle $\varpi^*T^*\tilde{X}$ defined by $\gamma^*\alpha_Y$ and by s_0

the zero section. Let (ζ_i) be the fibre coordinates of $\varpi^*T^*\tilde{X}$ with respect to the local coordinates $\left(\xi_1, \frac{\lambda_i}{\lambda_1}, \eta_a\right)$, then the section s is expressed as

$$\zeta_1 = \eta_{a0\cdots0} + \sum\nolimits_{\alpha_1 < a} \eta_{\alpha_1\cdots\alpha_n}\left(\frac{\lambda_2}{\lambda_1}\right)^{\alpha_2} \cdots \left(\frac{\lambda_n}{\lambda_1}\right)^{\alpha_n},$$

$$\zeta_i = \xi_1 \sum\nolimits_{k\geq 2, \alpha_k \geq 1} \eta_{\alpha_1\cdots\alpha_n}\alpha_k\left(\frac{\lambda_2}{\lambda_1}\right)^{\alpha_2} \cdots \left(\frac{\lambda_k}{\lambda_1}\right)^{\alpha_k - 1} \cdots \left(\frac{\lambda_n}{\lambda_1}\right)^{\alpha_n} \quad \text{for} \quad i \geq 2 .$$

The restriction of the induced bundle on $Y \times \mathbb{P}^{n-1}$ of $T^*\mathbb{P}^{n-1}$ to $\tilde{X}$ may be considered to be a subbundle of $T^*\tilde{X}$, which we denote by the same letter $T^*\mathbb{P}^{n-1}$. The quotient bundle $\varpi^*T^*\tilde{X}/\varpi^*T^*\mathbb{P}^{n-1}$, where $\varpi : \varphi^*T^*Y \to \tilde{X}$ is the canonical projection, is a line bundle and the image of the section $\gamma^*\alpha_Y$ of the bundle $\varpi^*T^*\tilde{X}$ to $\varpi^*T^*\tilde{X}/\varpi^*T^*\mathbb{P}^{n-1}$ corresponds to the divisor defined by the equation

$$\eta_{a0\cdots0} + \sum_{\alpha_1 < a} \eta_{\alpha_1\cdots\alpha_n}\left(\frac{\lambda_2}{\lambda_1}\right)^{\alpha_2} \cdots \left(\frac{\lambda_n}{\lambda_1}\right)^{\alpha_n} = 0 ,$$

which is a hypersurface of φ^*T^*Y and denoted by S. The restriction $s|S$ factors through the subbundle $\varpi^*T^*\mathbb{P}^{n-1}|S$. Denote by τ the section of $\varpi^*T^*\mathbb{P}^{n-1}|S$ defined by $s|S$ and τ_0 the zero section. We have the following lemma.

LEMMA. *Let $p: E \to X$ be a fibre bundle of rank r on a complex manifold of dimension n and F a subbundle of E of rank $r-1$. Let s be a section of E such that the restriction of s to a hypersurface S of X factors through $F|S$. Let τ be the section of $F|S$ defined by $s|S$. Denote by s_0 (resp. τ_0) the zero section of E (resp. $F|S$). We assume that no irreducible component of the normal cone C to $s(X) \cap s_0(X)$ in $s(X)$ is contained in F. Then we have $\mathbb{Z}[s] = \mathbb{Z}[\tau]$.*

Proof. Denote by $\mathscr{I}_s$ (resp. $\mathscr{I}_{s_0}$) the defining ideal of the analytic subspace $s(X)$ (resp. $s_0(X)$) in $\mathscr{O}_E$. Put $\mathscr{A} = \sum_{k\geq 0} \mathscr{I}_{s_0}^k/\mathscr{I}_{s_0}^{k+1}$ and $\mathscr{A}_C = \sum_{k\geq 0}(\mathscr{I}_s^k + \mathscr{I}_{s_0})/(\mathscr{I}_s^{k+1} + \mathscr{I}_{s_0})$. By definition $C = \operatorname{Specan} \mathscr{A}_C$. Let $\Phi : \mathscr{A} \to \mathscr{A}_C$ be the canonical morphism, which is surjective. The cone C is considered to be an analytic subspace of $\operatorname{Specan}\mathscr{A} = E$. Identifying X with $s_0(X)$, we consider the analytic subspace Z defined by the ideal $\mathscr{I}_s + \mathscr{I}_{s_0}$ to be the subspace of X. Then since $\mathscr{I}_s + \mathscr{I}_{s_0} \subset \operatorname{Ker}\Phi$, the analytic space C is an analytic subspace of $E|Z$. Let $C_1, \cdots$ be the irreducible components of C and m_i the multiplicity of C_i in C. Put $[C] = \sum m_i C_i$. Note that C is purely n dimensional. The cycle $[C]$ may be considered to be an element of the Borel-Moor homology group $H_{2n}(E|Z)$. The projection $p|Z : E|Z \to Z$ induces the isomorphism of the homology groups $(p|Z)^* : H_{2(n-r)}(Z) \to H_{2n}(E|Z)$ and the localized top Chern class $\mathbb{Z}[s]$ is defined to be $((p|Z)^*)^{-1}([C])$. Denote by $q : F|Z \to Z$ the projection and by $j : F|Z \to E|Z$ the injection. Then by Fulton [4] and the assumption it is easy to see that $(q^*)^{-1}([j^*C]) = (q^*)^{-1}(j^*[C]) = ((p|Z)^*)^{-1}([C])$, where j^*C is the pull back of the analytic subspace C. Let $\mathscr{I}_S$ (resp. $\mathscr{I}_F$) be the defining ideal of the hypersurface S (resp. the subbundle F) in $\mathcal{O}_X$ (resp. $\mathscr{A}$). By assumption $\mathscr{I}_S \subset \mathscr{I}_s + \mathscr{I}_{s_0}$. The morphism Φ induces the morphism $\bar{\Phi}: \mathscr{A}/(\mathscr{I}_S\mathscr{A} + \mathscr{I}_F) \to \mathscr{A}_C/\Phi(\mathscr{I}_F)$. We have Specan $\mathscr{A}/(\mathscr{I}_S + \mathscr{I}_F) = F \,|\, S$ and Specan $\mathscr{A}_C/\bar{\Phi}(\mathscr{I}_F) = j^*C$ is the normal cone $C_{\tau(S) \cap \tau_0(S)}(\tau(S))$ to $\tau(S) \cap \tau_0(S)$ in $\tau(S)$. Therefore we have

$$\mathbb{Z}[\tau] = (q^*)^{-1}([C_{\tau(S) \cap \tau_0(S)}(\tau(S))]) = (q^*)^{-1}(j^*[C]) = ((p \,|\, Z)^*)^{-1}([C]) .$$

By the lemma we have $\mathbb{Z}[s] = \mathbb{Z}[\tau]$. The section τ of the bundle $\varpi^*T^*\mathbb{P}^{n-1} \,|\, S$ is expressed as

$$\zeta_i = \xi_1 \sum_{k \geq 2, \alpha_k \geq 1} \eta_{\alpha_1 \cdots \alpha_n} \alpha_k \left(\frac{\lambda_2}{\lambda_1}\right)^{\alpha_2} \cdots \left(\frac{\lambda_k}{\lambda_1}\right)^{\alpha_k - 1} \cdots \left(\frac{\lambda_n}{\lambda_1}\right)^{\alpha_n} \qquad \text{for} \quad i \geq 2 .$$

The divisor Θ in S defined by the equation $\xi_1 = 0$ is the restriction of the inverse image by ϖ of the exceptional divisor of the map $\tilde{X} \to X$. The bundle $\varpi^*T^*\mathbb{P}^{n-1} \,|\, S \otimes \mathcal{O}(-\Theta)$ has the section τ' defined by

$$\zeta_i = \sum_{k \geq 2, \alpha_k \geq 1} \eta_{\alpha_1 \cdots \alpha_n} \alpha_k \left(\frac{\lambda_2}{\lambda_1}\right)^{\alpha_2} \cdots \left(\frac{\lambda_k}{\lambda_1}\right)^{\alpha_k - 1} \cdots \left(\frac{\lambda_n}{\lambda_1}\right)^{\alpha_n} \qquad \text{for} \quad i \geq 2 ,$$

and by *Residual formula for top Chern classes* of Fulton [4] we have

$$\mathbb{Z}[\tau] = \mathbb{Z}[\tau'] + \sum_{i=1}^{n-1} (-1)^{i-1} c_{n-i-1}(T^*\mathbb{P}^{n-1}) \cap \Theta^{i-1} \cdot [\Theta] ,$$

from which we infer readily that

$$ChR\varphi_* \mathcal{O}_{\tilde{X}} = T^*_X Y + (-1)^n \sum (-p)^{i-1} p \binom{n}{n-1-i} T^*_0 Y .$$

Therefore we have

$$Eu_X(o) = (-1)^n n - \sum (-p)^{i-1} p \binom{n}{n-1-i} .$$

Example 4. Let σ, τ be automorphisms of $\mathbb{C}^3$ defined to be

$$\sigma: (x_1, x_2, x_3) \to (e^{2\pi i/3} x_1, e^{2\pi i/3} x_2, e^{2\pi i/3} x_3)$$

$$\tau: (x_1, x_2, x_3) \to (-x_1, -x_2, x_3) .$$

We consider the quotient space $X = \mathbb{C}^3/(\sigma, \tau)$. The complex space X may be considered to be a subvariety of $Y = \mathbb{C}^{12}$ with coordinate $(y_\alpha, z_\alpha, w_\alpha)$, where the suffix α of y_α runs through $(\alpha_1, \alpha_2, \alpha_3)$ such that $\alpha_i \geq 0$ $\sum \alpha_i = 3$, $\alpha_1 + \alpha_2 \equiv (2)$, the suffix α of z_α runs through $(\alpha_1, \alpha_2, \alpha_3)$ such that $\alpha_i \geq 0$, $\sum \alpha_i = 3$, $\alpha_1 + \alpha_2 \equiv (1)$ and the suffix α of w_α is either $(1, 1, 4)$ or $(3, 3, 0)$, by the injection map

$$X = \mathbb{C}^3/(\sigma, \tau) \ni \text{class of } (x_i) \longrightarrow (y_\alpha, z_\alpha, w_\alpha) \in \mathbb{C}^{12}$$

$$\begin{cases} y_{(\alpha_1,\alpha_2,\alpha_3)}=x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\,, \\ z_{(\alpha_1,\alpha_2,\alpha_3)}=x_1^{2\alpha_1}x_2^{2\alpha_2}x_3^{2\alpha_3}\,, \\ w_{(\alpha_1,\alpha_2,\alpha_3)}=x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\,. \end{cases}$$

Put $\tilde{X}=\{\lambda_j^a\xi_i=\lambda_i^a\xi_j\}\subset\mathbb{C}^3\times\mathbb{P}^{12}$ and define the map

$$\varphi:\tilde{X}\ni(\xi,\lambda)\longrightarrow(y_\alpha,z_\alpha,w_\alpha)=\varphi(\xi,\lambda)\in X\subset\mathbb{C}^{12}$$

to be

$$y_{(\alpha_1,\alpha_2,\alpha_3)}=\xi_3\left(\frac{\lambda_1}{\lambda_3}\right)^{\alpha_1}\left(\frac{\lambda_2}{\lambda_3}\right)^{\alpha_2},$$

$$z_{(\alpha_1,\alpha_2,\alpha_3)}=\xi_3^2\left(\frac{\lambda_1}{\lambda_3}\right)^{2\alpha_1}\left(\frac{\lambda_2}{\lambda_3}\right)^{2\alpha_2},$$

$$w_{(\alpha_1,\alpha_2,\alpha_3)}=\xi_3^2\left(\frac{\lambda_1}{\lambda_3}\right)^{\alpha_1}\left(\frac{\lambda_2}{\lambda_3}\right)^{\alpha_2}.$$

The map $\varphi:\tilde{X}\to X$ is generally 2-fold, the surface Θ defined by the equation $\xi_3=0$ contracts to the point $o\in X\subset Y$ and the curve in $\tilde{X}$ defined by the equations $\lambda_1=\lambda_2=0$ is mapped biholomorphically to the line $L\subset\mathbb{C}^{12}$ defined by the equations $y_\alpha=0$ for $(\alpha)\neq(0,0,3)$, $z_\alpha=0$ and $w_\alpha=0$. In a similar manner to Example 3 we have

$$ChR\varphi_*\mathcal{O}_{\tilde{X}}=2T_X^*Y+T_L^*Y+(-1)^3\sum_{i=1}^{2}(-3)^{i-1}3\binom{3}{3-1-i}T_o^*Y,$$

from which we have

$$Eu_X(o)=-2\,.$$

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